Proof

Step 1: What are we trying to prove?

The determinant of a matrix is a sum containing every possible product of items in different rows and columns, each attached to some sign.

Ex:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

If we pick a set of cells, all in different rows and columns, we can find their signed product in the determinant. For example, if we want to find the sign attached to b, f, g in the determinant above, we can just look at the second component of the sum:

$$-b\begin{vmatrix} d & f \\ g & i \end{vmatrix}$$

We can now continue this process:

$$-b\begin{vmatrix} d & f \\ g & i \end{vmatrix} = -b\cdot[d\cdot|i| - f\cdot|g|]$$

Extracting the component containing f, we get

$$-f\cdot|g| = -fg$$

So

$-b\begin{vmatrix} d & f \\ g & i \end{vmatrix}$ gives us $-b\cdot[-f\cdot g] = bfg$,

Which means that the determinant attaches a positive sign to this product, when evaluated by the following procedure:

For each element in row 1:

1. Multiply the element by its sign and by the determinant of the matrix not in its row or column.

2. Add that number to a sum.

The final result of this sum is the determinant, by this definition.

We will now attempt to prove that we can use any row for this procedure, not just row 1.

Step 2: Going from an $N \times N$ matrix to an $(N-1)\times(N-1)$ matrix

Suppose we wanted to find the sign attached to $b \cdot f \cdot g$ in our old 3×3 matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

This time, let's attempt it by using the second row

$$-d\cdot\begin{vmatrix} b & c \\ h & i \end{vmatrix} + e\cdot\begin{vmatrix} a & c \\ g & i \end{vmatrix} - f\cdot\begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

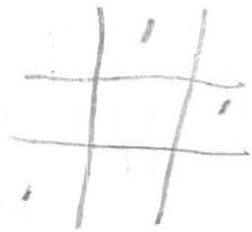
Our signed product will come from the third term.

Now we need to figure out how removing f from the matrix changes its architecture.

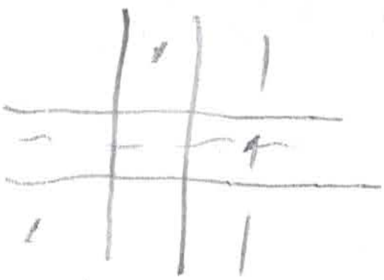
From here, we'll be using dots to represent where our numbers are in our matrices. It should be easier to follow than writing out every item in each matrix.

Here's what our matrix looks like now:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$



By eliminating the row and column containing f, we get the following arrangement of the factors whose signs we are looking for



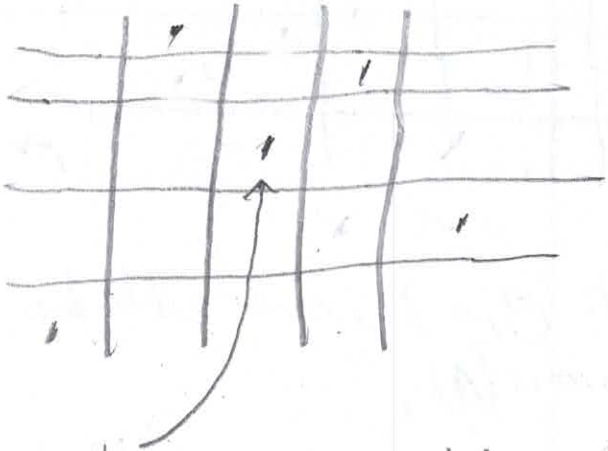
For a matrix of dots, define its "output sign" to be the sign of the product of the values in the locations of the dots in the determinant.

Ex: OS() = +1

OS() = −1

* OS = output sign

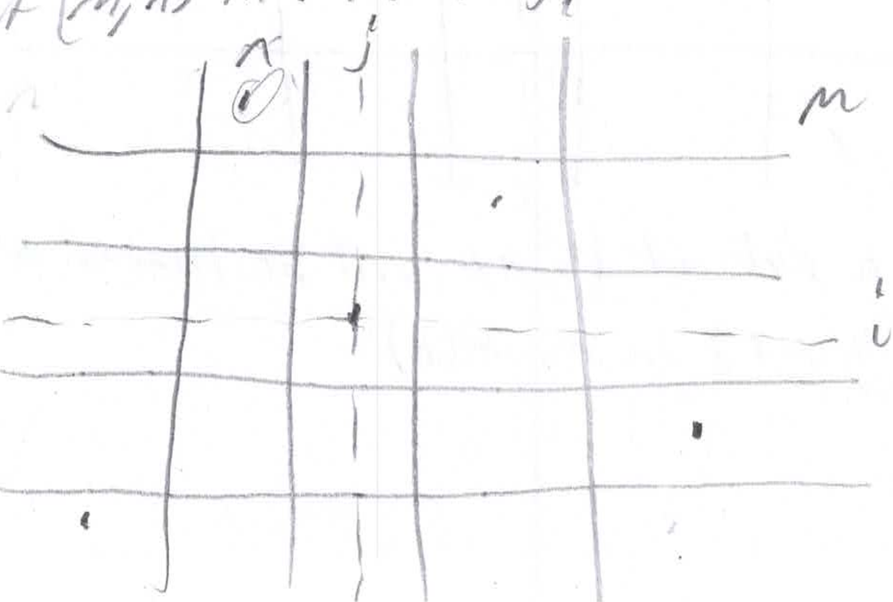
Now let's try a more ambiguous example to see just where our dots end up when we reduce the dimensions of our matrix.



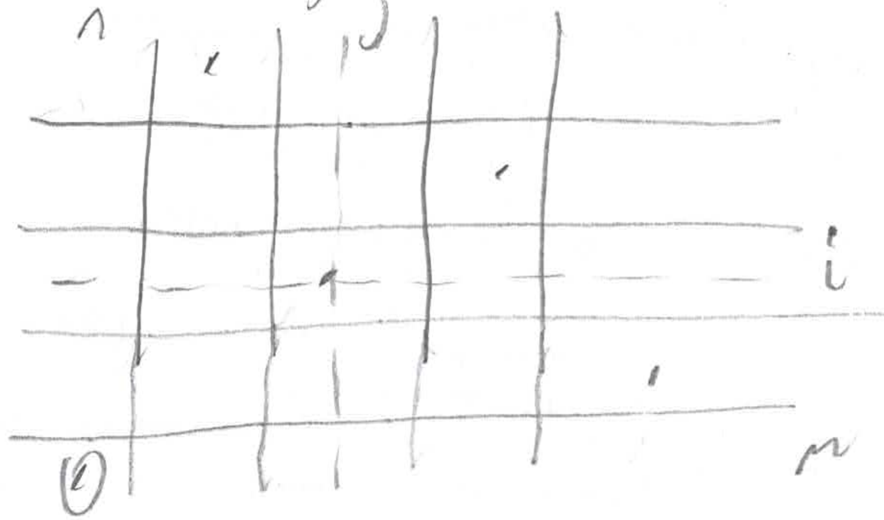
We will call this matrix A

Call this value $a_{i,j}$. We will use row i to compute the determinant. Let's figure out the architecture of the matrix whose determinant we need to multiply by $a_{i,j} \cdot (-1)^{i+j}$, also known as the matrix we use to compute the minor of $a_{i,j}$.

If $m<i$ and $n<j$, then a dot at (m,n) in A will be at (m,n) in minor(A).

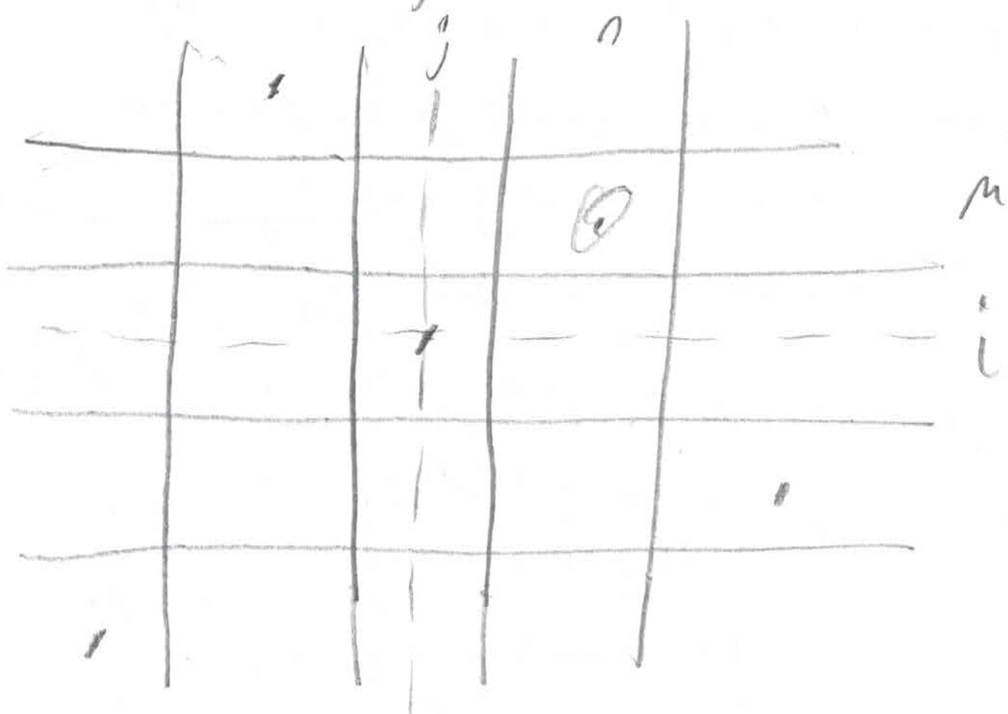


If $m > i$ and $n < j$:



then a dot at (m, n) in a will be at $(m-1, n)$ in minor(A).

If $m < i$ and $n > j$:



a dot at (m, n) will be located at $(m, n-1)$ in minor(A)

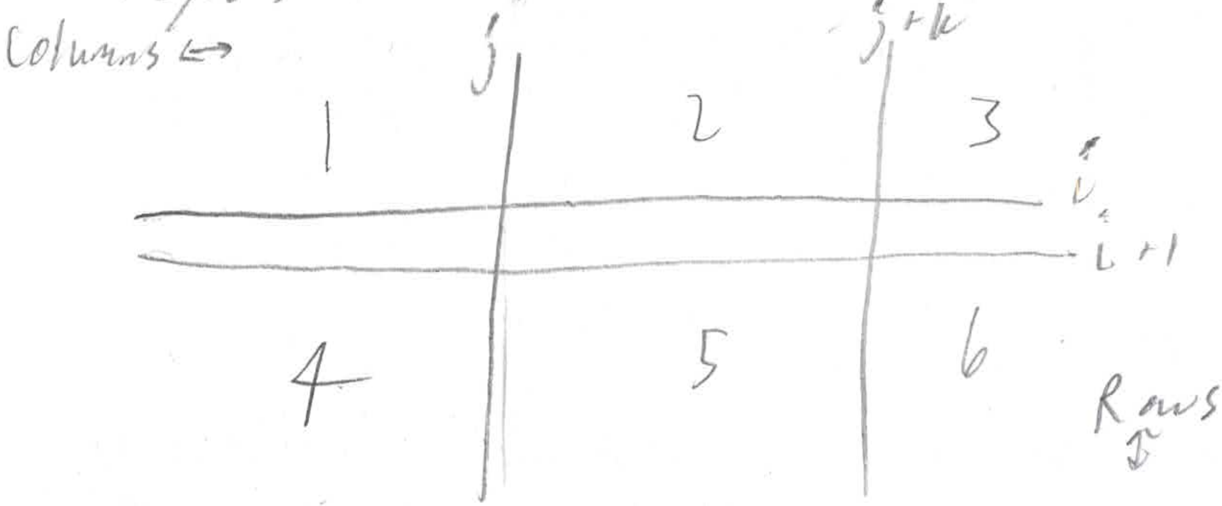
If $m > i$ and $n > j$:

Then a dot at (m, n) in A will be at $(m-1, n-1)$ in minor A. Note that $m \neq i$ and $n \neq j$ because each dot must be in a distinct row and column

Step 3: The Setup

Here's how we can use this to prove that the determinant can be calculated from any row. We can examine the difference between the minor of $a_{i,j}$ and $a_{i+1,j+k}$, where $a_{i,j}$ is one of our dots and $a_{i+1,j+k}$ is the dot in the next row. If these two items are attached to the same sign, the output signs of their minors should be equal. If not, they should be opposites. This is because the final sign of the product in the determinant should always be the same, regardless of the row used to compute it.

Now that we have a clear link between what we've examined and the conclusion we want to prove, we can start the substantive part of this proof. We can split our matrix A into regions as shown below.



In regions 1, 3, 4, and 6, our previous work shows that $a_{m,n}$ maps to the same point in both minors.

$(m,n) \to (m,n)$ in region 1

$(m,n) \to (m,n-1)$ in region 3

$(m,n) \to (m-1,n)$ in region

$(m,n) \to (m-1,n-1)$ in region 6

In regions 2 and 5, things are slightly different. In region 2, $(m,n) \to (m,n-1)$ in minor$(a_{i,j})$.

However, point (m,n) in region 2 maps to (m,n) in $minor(a_{i+1,j+k})$. In region 5, $(m,n) \to (m-1,n-1)$ in $minor(a_{i,j})$ and $(m,n) \to (m-1,n)$ in $minor(a_{i+1,j+k})$. In addition, $minor(a_{i,j})$ has a dot at $(i,j+k-1)$ which has value $a_{i+1,j+k}$. This is because the dot in our original A at $(i+1,j+k)$ maps to that point in $minor(a_{i,j})$. Similarly, $a_{i,j}$ is in cell (i,j) in $minor(a_{i+1,j+k})$.

Here's what this tells us:

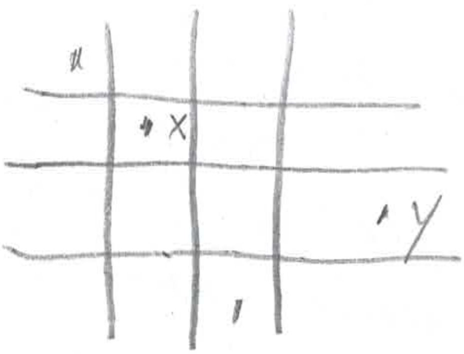
- All points in regions 1, 3, 4, 6 in A remain unchanged in both minors. This means that the two minors should be identical. for all $n<j$, the two minors should be identical. This also means that the two minors should be identical for all $n \geq j+k$ in A, which would translate to $n \geq j+k-1$ in both of the minors. So, both minors are identical except for when $j \leq n \leq j+k-1$.

- When $j \leq n < j+k-1$, $a_{m,n}$ in $minor(a_{i,j})$ can be moved to its location in $minor(a_{i+1,j+k})$ by moving one space to the right to $a_{m,n+1}$.

Our special case is the points $a_{i,j}$ and $a_{i+1,j+k}$ from our original matrix A. This gives a point at $(i, j+k-1)$ in minor$(a_{i,j})$ and (i,j) in minor$(a_{i+1,j+k})$.

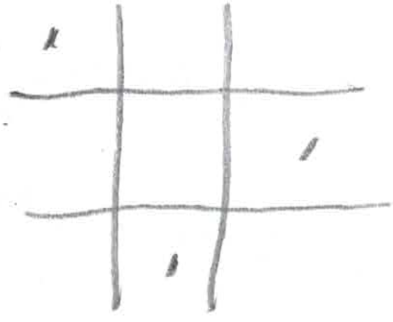
So, every point in our minor in columns j through $j+k-1$ gets moved one space to the right and maintains its row, except the point in column $j+k-1$, which loops back to column j and maintains its row.

Here's an example of how this process looks:

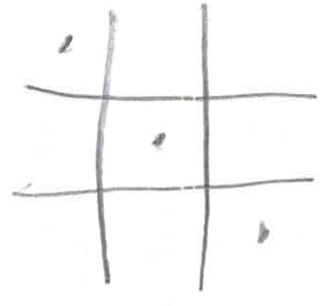


Comparing the minors from points x and y we can see the following.

Minor(x)



minor(y)



We can also determine minor(y) by executing a "loop" procedure on minor(x), from row 2 to row 3

Now, we need to find the sign our product will be attached to in the final determinant calculation. Measuring from point $a_{i,j}$ in our matrix A, we get $(-1)^{i+j} \cdot OS(minor(a_{i,j}))$.

Measuring what we hope to be the same thing from point $a_{i+1,j+k}$ gives $(-1)^{(i+1)+(j+k)} \cdot OS(minor(a_{i+1,j+k}))$

$$(-1)^{i+j} \cdot OS(minor(a_{i,j})) = (-1)^{i+j+k+1} \cdot OS(minor(a_{i+1,j+k}))$$

so, $$OS(minor(a_{i,j})) = -1^{k+1} \cdot OS(minor(a_{i+1,j+k}))$$

In other words, if k (the size of our "loop") is even, the output signs of the two minors should be opposites. Otherwise they should be the same.

Now, we've reduced the problem.

To show that any $N \times N$ matrix can have its determinant calculated from any row, we need to show that any $(N-1) \times (N-1)$ matrix switches its output sign when an even-sized "loop" is executed on it, and remains the same after an odd-sized "loop".

Step 4: Induction

Let $g(N)$ be the statement: Any $N \times N$ matrix can have its determinant computed from any row.

Let $h(N)$ be the statement: Any $N \times N$ matrix has the property that executing a k-sized loop on it will change its output sign if k is odd and keep it the same if k is even.

From the statement at the top of this page, proven on page 11, we know that $g(N-1)$ implies $h(N)$.

We will now attempt to prove $g(N)$, assuming that $h(N)$ and $g(N-1)$ are true.

Firstly, we must recognize that our operation goes from column j to column $j+k-1$. (13)

Case 1: $j=1$ and $j+k-1$ is the width of the matrix, also known as N.

In this case, the size of our loop is N. Performing a loop moves every item to the right by one space, except for $a_{m,k}$, which is mapped to $a'_{m,1}$. Since we assume $h(N)$, we can use any row to calculate the output sign of our $N \times N$ matrix. We will use row m both times. Since $h(N)$ is true, we can run a calculation to extract an output sign from before and after the transformation. We will call our matrices A and A':

same as $j+k-1$ since $j=1$

We're just using "prime" to denote that the value is in the matrix after the transformation

$$OS(A) = (OS(minor(a_{m,k})) \cdot (-1)^{m+k}$$

$$OS(A') = OS(minor(a_{m,1})) \cdot (-1)^{m+1}$$

Note that $(a_{m,k}) = (a'_{m,1})$ and that the minor of those two points are the same, because every column other than column k is preserved and shifted in this transformation.

So:

$$OS(A) = X \cdot (-1)^{m+k}$$

and $$OS(A') = X \cdot (-1)^{m+1}$$

$$\frac{OS(A)}{OS(A')} = (-1)^{k-1}$$

$$OS(A) = OS(A') \cdot -1^{k-1}$$

This means that the output signs are different if k is even and equal if k is odd.

Case 2: $j \neq 1$ or $j+k-1 \neq N$

In this case, there always exists a point in either column 1 or N. Since we assume $k(N)$, the output sign of our matrix can be calculated using any row. We will once again call our pre-transformation matrix A, and our post transformation matrix A'.

If $j \neq 1$:

• We denote the loop function as L. Ex: $A' = L(A)$

$$OS(A) = (-1)^{m+1} \cdot OS(\text{minor}(a_{m,1}))$$

$$OS(A') = (-1)^{m+1} \cdot OS(L(\text{minor}(a_{m,1})))$$

SO,

$$\frac{OS(A)}{OS(A)} = \frac{OS(L(minor(a_{m,1})))}{OS(minor(a_{m,1}))}$$

If k is odd, the fraction on the right will evaluate to 1 by the assumption that $g(N-1)$ is true. If k is even it is equal to -1 by the same assumption. Therefore, if k is odd, the output sign must be preserved, and if k is even, the output sign must change.

If $j=1$ and $j+k-1 \neq N$, the exact same argument can be made using some point $a_{m,N}$ to compute the determinant.

Step 5: The Punch Line

Now that we've shown both possible cases, we've proven that $g(N-1)$ and $h(N)$ being true implies that $g(N)$ is also true. This in turn implies that $h(N+1)$ is true as well. Now, it's been proven that

$g(N-1)$ and $h(N)$ imply $g(N)$ and $h(N+1)$

Now, once we prove the base cases, we can show that $g(N)$, and more importantly $h(N)$, are true for all integers N.

Proving $g(1)$:

The only size of a loop that we can execute on a 1×1 matrix is 1. This preserves the sign of the only number in the matrix in the determinant. It is always positive by definition.

Proving $h(2)$:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot |d| - b \cdot |c| = ad - bc$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -c \cdot |b| + d \cdot |a| = ad - bc$$

With our base cases proven, we can now claim that the determinant can indeed be computed using any row in a matrix.

QED